

The two foci property of ellipses

In Unit 4, Section 3.4 you saw that if F_1 and F_2 are the foci of an ellipse whose major axis has length $2a$, then $PF_1 + PF_2 = 2a$ for every point P on the ellipse.

The unit stated that the ‘reverse’ of this fact is also true. That is, the following fact holds.

Suppose that F_1 and F_2 are any two points in the plane, and a is any positive number. Then the curve formed by the points P that satisfy the equation

$$PF_1 + PF_2 = 2a,$$

is the ellipse with foci F_1 and F_2 and major axis of length $2a$.

This fact is the basis of the gardener’s method for drawing an ellipse.

As mentioned in Unit 4, you can see intuitively that this fact is true, as follows. Suppose that, as in the box, F_1 and F_2 are any two points in the plane, and a is any positive number. Consider the ellipse with foci F_1 and F_2 and major axis of length $2a$. You know that every point on this ellipse satisfies the equation $PF_1 + PF_2 = 2a$. So the points that satisfy the equation $PF_1 + PF_2 = 2a$ are all the points on the ellipse, and possibly some further points. However, if you think about carrying out the gardener’s method, then you can see that the points that satisfy the equation $PF_1 + PF_2 = 2a$ certainly form an ellipse-like curve. So, since this ellipse-like curve contains the ellipse, it must actually be the ellipse – there are no points on it other than the points on the ellipse.

We can prove the fact in the box formally as follows. Suppose that, as in the box, F_1 and F_2 are any two points in the plane, and a is any positive number. We can introduce coordinate axes in the plane, positioning them to ensure that F_1 and F_2 lie on the x -axis, with F_1 on the positive part of the x -axis, F_2 on the negative part, and the y -axis halfway between them, as shown in Figure 1.

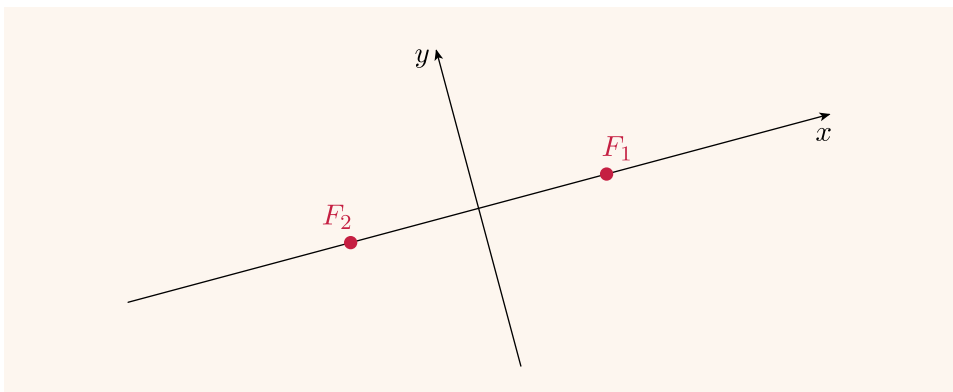


Figure 1 Two points F_1 and F_2 and suitably-chosen coordinate axes

Now consider the curve formed by the points P that satisfy the equation $PF_1 + PF_2 = 2a$, as illustrated in Figure 2.

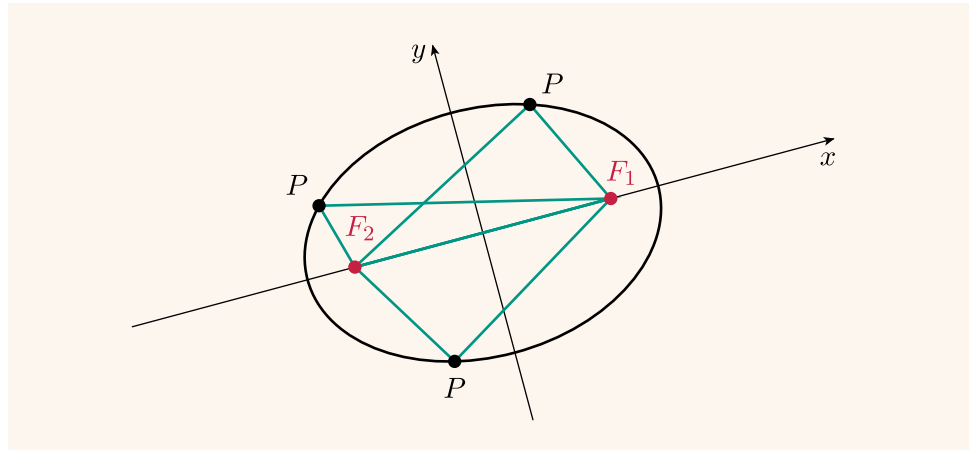


Figure 2 The curve with equation $PF_1 + PF_2 = 2a$

First consider the particular point P where this curve crosses the positive part of the x -axis, as shown in Figure 3. Suppose that this point has x -coordinate p . Then the equation $PF_1 + PF_2 = 2a$ gives

$$(p - f) + (p + f) = 2a,$$

where f is the distance of each of F_1 and F_2 from the origin. Hence $2p = 2a$, which gives $p = a$.

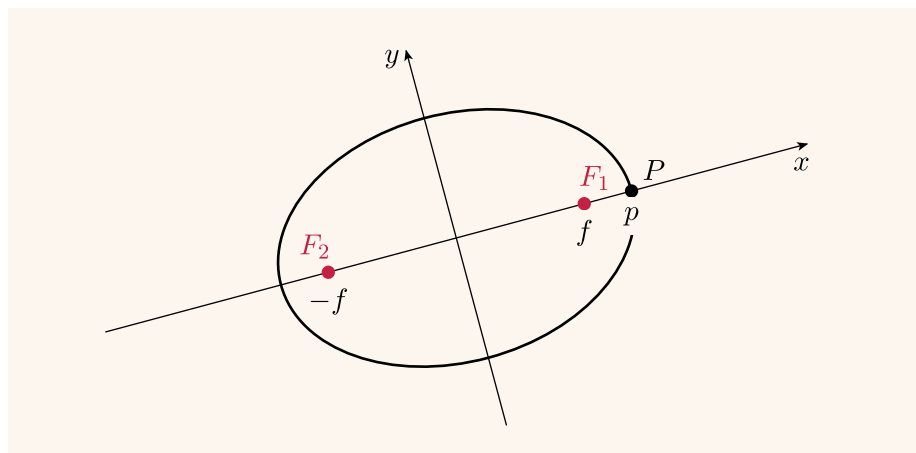


Figure 3 The point P where the curve crosses the positive part of the x -axis

So the curve crosses the x -axis at $(a, 0)$, and hence, by symmetry, also at $(-a, 0)$. Let e be the number such that the x -coordinate of F_1 is ae . Then the situation is as shown in Figure 4. Notice that we must have $0 < e < 1$.

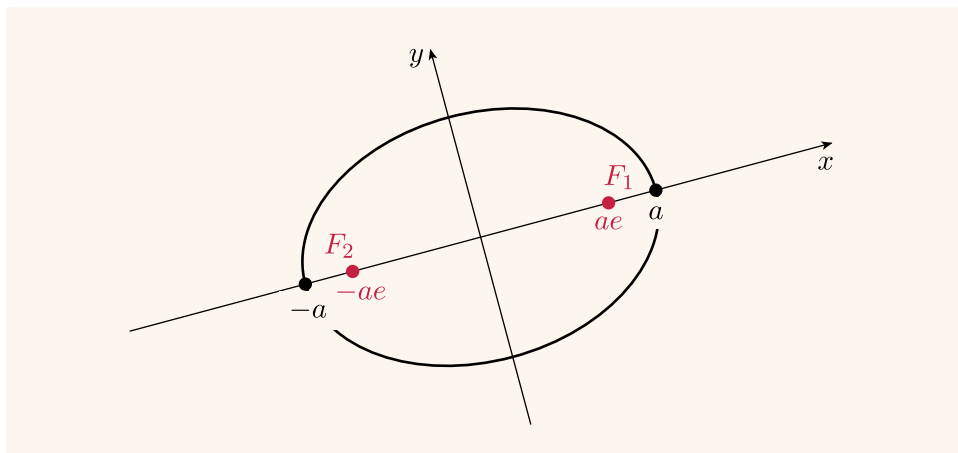


Figure 4 The x -coordinates of the points where the curve crosses the x -axis, and the x -coordinates of F_1 and F_2

Now consider a general point P on the curve, as illustrated in Figure 5. Let r be the distance PF_1 . Then, since $PF_1 + PF_2 = 2a$, we have $PF_2 = 2a - r$. Also, let x be the x -coordinate of the foot of the perpendicular from P to the x -axis (so x could be positive, negative or zero).

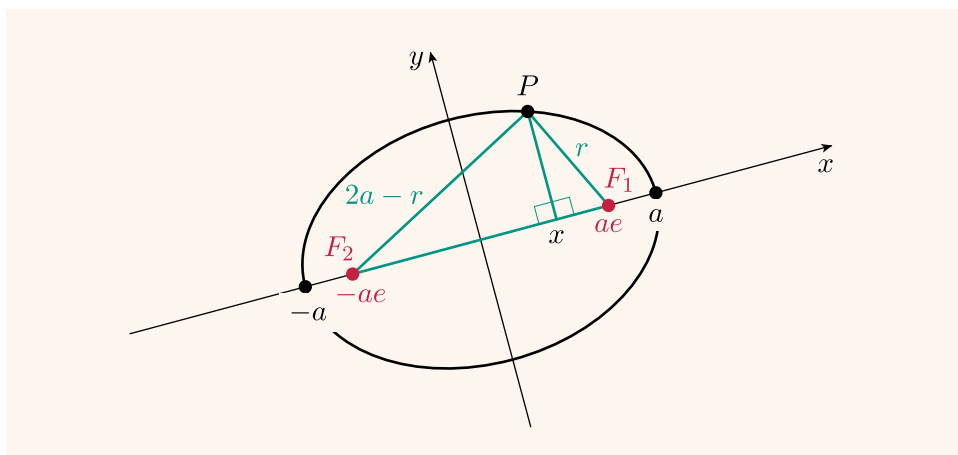


Figure 5 A point P on the curve with equation $PF_1 + PF_2 = 2a$

By applying Pythagoras' theorem to each of the two right-angled triangles in Figure 5, and using the fact that they have a side in common, we obtain the equation

$$r^2 - (ae - x)^2 = (2a - r)^2 - (ae + x)^2.$$

Multiplying out in this equation and simplifying gives

$$\begin{aligned} r^2 - (a^2e^2 - 2aex + x^2) &= (4a^2 - 4ar + r^2) - (a^2e^2 + 2aex + x^2) \\ r^2 - a^2e^2 + 2aex - x^2 &= 4a^2 - 4ar + r^2 - a^2e^2 - 2aex - x^2 \\ 4ar &= 4a^2 - 4aex \\ r &= a - ex. \end{aligned}$$

This equation can be written as

$$r = e \left(\frac{a}{e} - x \right).$$

Now let d be the line with equation $x = a/e$, which lies further along the positive part of the x -axis than the curve, since $0 < e < 1$. Then the expression $a/e - x$ in the equation above is the distance between P and d , as shown in Figure 6. Since also r is the distance between P and F_1 , the equation above can be written as

$$PF_1 = e Pd.$$

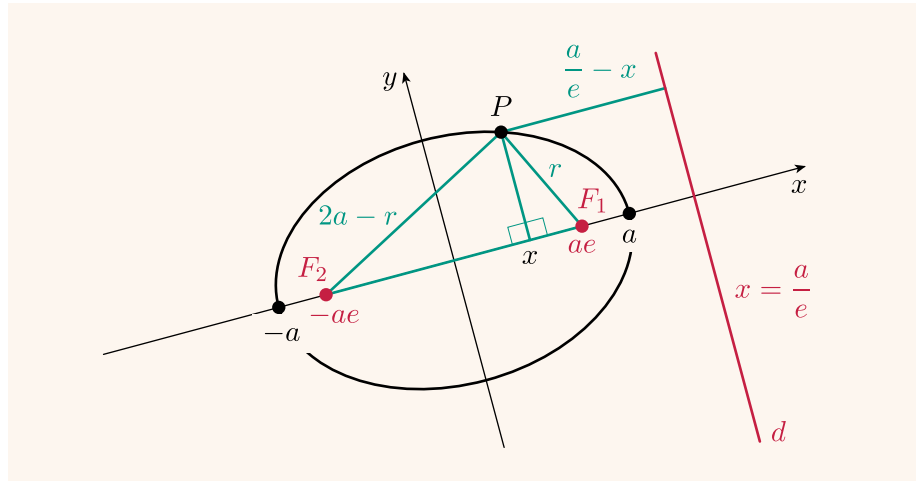


Figure 6 The line d with equation $x = a/e$

This tells us that all the points that satisfy the equation $PF_1 + PF_2 = 2a$ also satisfy the equation $PF_1 = e Pd$, which, as you know, is an equation that specifies the ellipse with focus F_1 , directrix d and eccentricity e . So all the points that satisfy the equation $PF_1 + PF_2 = 2a$ lie on this ellipse. You've already seen that all the points that lie on this ellipse satisfy the equation $PF_1 + PF_2 = 2a$. So the two curves are the same. This proves the fact in the box, since, as you have seen in Unit 4, the ellipse with equation $PF_1 = e Pd$ has foci F_1 and F_2 and major axis of length $2a$.